A Logical Notation With (Only) Two Primitive Signs.[1]

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I. Two Syntactical Systems.

I set out two syntactical systems for logic *cum* set theory, <*I share Quine's post-1960 preference for excluding set theory from the scope of logic*> each using only two primitive signs, namely the left and right parenthesis. These syntaxes are adequate to express the system ML of Quine (1951), hence they suffice to express any statement involving quantification and set membership. The second system eliminates the explicit notation for quantifiers from the first system, by allowing a name for the domain to appear in atomic formulae.

Let Greek letters be metalogical. A *term* is either a variable or the name of an individual. The first system has the following formation rules:

I. *Primitive signs*: '(' ')'
II. *Symbols* (well-formed signs):
- A. Atomic:
  1. '()' (hereinafter abbreviated as 'o') and '(o)' are atomic symbols.
  2. If $\ulcorner(\sigma)\urcorner$ is an atomic symbol, $\ulcorner(\sigma o)\urcorner$ is an atomic symbol. (The atomic symbol σ is a *variable* iff the number of instances of 'o' in σ is even and exceeds 2.)
- B. Molecular:
  1. If ρ and σ are symbols, $\ulcorner\rho\sigma\urcorner$ is a symbol.
  2. If σ is a symbol, $\ulcorner(\sigma)\urcorner$ is a symbol.

III. *Formulae*:
- A. Atomic: If α and β are terms, $\ulcorner((o)(\alpha\beta))\urcorner$ is an atomic formula.
- B. Molecular:
  1. If ϕ is a formula and α is a term, $\ulcorner((\alpha)\phi)\urcorner$ is a formula.
  2. If ϕ and ψ are formulae, then $\ulcorner((\phi)(\psi))\urcorner$ is a formula.

Quine's (1951: 80) variables *w*, *x*, *y*, *z*, *w'*,... may abbreviate the variables defined in II.A.2, with Quine's alphabetic order correlated with the order of generation of those variables. Formulae and variables as defined above are collectively known as *expressions*. The *minimum depth* of an expression is a crucial concept: it is the shallowest instance of 'o' in the expression, i.e., the smallest number of pairs of parentheses enclosing any instance of 'o' in the expression. An unabbreviated variable has minimum depth 1.

The system ML consists of first order logic and set theory (the latter axiomatized in an elegant yet idiosyncratic manner), and builds on a mere three primitives: a predicate, set membership; the universal quantification of variables; and the stroke functor, expressing alternate denial. The minimum depth of an unabbreviated formula provides an effective criterion for determining whether an

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expression is a variable, atomic formula, quantified formula, or an application of the stroke functor. Here's how the above notation suffices to express these primitives:

- If '$\in$' is taken as an abbreviation of '(o)', the *atomic formulae* described in rule III.A may be abbreviated as $\ulcorner(\in(\alpha\beta))\urcorner$, meaning "$\alpha$ is a member of β", where α and β are terms, and further abbreviated to yield Quine's atomic formula $\ulcorner\alpha\in\beta\urcorner$. Unabbreviated atomic formulae have minimum depth 2;
- $\ulcorner((\alpha)\phi)\urcorner$ in rule III.B.1 may be taken as a *universally quantified* formula, and abbreviated as $\ulcorner(\forall\alpha(\phi))\urcorner$. Unabbreviated quantified formulae in our notation have minimum depth 3;
- The conventional notation for the *stroke truth functor* is the down-arrow '$\downarrow$'. A formula of the form '$((\phi)(\psi))$', stipulated by rule III.B.2, denotes an instance of this functor and so may be abbreviated as '$(\phi\downarrow\psi)$'. An unabbreviated instance of the stroke functor has minimum depth 4.

The rules in III define formulae that are syntactic equivalents of the formulae (Quine, p. 124f) possible under ML. The latter, in turn, suffice to define abstraction, identity, ordered pair, relation, function, number, etc. (pp. 323-24), and to express any purely mathematical statement.

The second system of notation simply replaces the rules in III with the following:

III′. *Formulae*:

A. If α is a term, and β is either a term or '(oo)', then $\ulcorner((o)(\alpha\beta)))\urcorner$ is an atomic formula.
B. Molecular:
   1. If ϕ is a formula, then $\ulcorner(\phi)\urcorner$ is a formula.
   2. If ϕ and ψ are formulae, then $\ulcorner(\phi\psi)\urcorner$ is a formula.

III′ incorporates two departures from III. First, III′.A eliminates III.B.1 by allowing the name '(oo)' to appear in place of 'β' in '$\alpha\beta$'. '(oo)' can be interpreted as denoting the class of extant things, assumed nonempty, and will be abbreviated by 'E'. <*Note that I call* 'E' *a* name, *not a* constant. *How does 'E' differ from what is conventionally termed the* domain? *Does* 'E' *name the set whose members are all the elements of the domain, and is this set identical to Quine's universal set* V? *If not, I do not understand what is perhaps your paper's most original detail. And why is* 'E' *prefixed*?> The presence of 'E' effectively distinguishes a quantified formula from an atomic one. The other departure from the first system replaces III.B.2 with III′.B.1–2. Semantically speaking, this amounts to replacing the stroke functor with the expressively adequate pair denial and conjunction. A minimum depth of 3 now signals that a formula is truth functional rather than quantificational. The second departure is optional and serves mainly to make the notation more economical.[2]

Letting, as before, ϕ and ψ be metalogical names for formulae and α and β for terms, the second notation may be translated into familiar symbolism as follows:

D1. $\ulcorner\alpha\in\beta\urcorner$ for $\ulcorner((o)(\alpha\beta))\urcorner$ <*Invoking the boundary logic interpretation of parens,* $((o)(\alpha\beta)) = ((\alpha\beta)) = \alpha\beta$.> eliminating '$\in$'

D2. $\ulcorner\alpha\in E\urcorner$ for $\ulcorner E\alpha\urcorner$ or $\ulcorner((o)(\alpha(oo)))\urcorner$ <*or* $\ulcorner\alpha(oo)\urcorner$>

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2. Those who advocate taking conjunction and denial as primitive truth functors are as distinguished as they are few, namely Rosser and Quine. I submit that among the binary functors, conjunction presents the fewest semantic ambiguities.

D3. ⌜¬ϕ⌝ for ⌜(ϕ)⌝
D4. ⌜ϕ∧ψ⌝ for ⌜(ϕψ)⌝
D5. ⌜ϕ→ψ⌝ for ⌜((ϕ(ψ)))⌝
D6. ⌜ϕ∨ψ⌝ for ⌜(((ϕ)(ψ)))⌝
D7. ⌜ϕ↔ψ⌝ for ⌜((ϕ(ψ))(ψ(ϕ)))⌝
D8. ⌜∀α(ϕ)⌝ for ⌜¬(Eα∧¬ϕ)⌝ or ⌜((Eα(ϕ)))⌝.

<An advantage of the second notation you do not mention is its economy. Making denial primitive can result in a substantial reduction in the length of formulae. Take the very simple formula '(a∈b) ∧ (c∈d)', *which translates into your first notation as* '(((((o)(ab)))(((o)(cd))))((((o)(ab)))(((o)(cd))))))'. *Your second notation reduces that to* '(((o)(ab))((o)(cd)))'.

Your D3–D7 reveal that, abstracting from the outermost parens, your notation is dual to the boundary notation I advocate and isomorphic to Peirce's alpha graphs. Below I list some equivalences between boundary and standard notation. The interpretation of the atomic symbols '()' *and* '(())' ['o' *and* '(o)' *in your notation*] *is the key to* boundary logic. *The boundary notation for* '(a∈b) ∧ (c∈d)' *is* 'abcd'. *While supremely economical, this has the drawback of suggesting commutativity and associativity in a context where they do not hold.*

The existential graphs have one more trick up their sleeve: a variable α can appear outside of an atomic formula. If there is another instance of α at the same depth that is part of an atomic formula, then the stand-alone instance is redundant. '(α(—α—))' *serves to toggle the quantification of* '—α—'. *The boundary perspective on quantification only strengthens the utility of making denial primitive.* >

Standard	*Boundary; Existential Graphs*	*Boundary; Entitative Graphs*
T,F	(()), ()	(), (())
¬ϕ	(ϕ)	
ϕ∨ψ, ϕ∧ψ	((ϕ)(ψ)), ϕψ	ϕψ, ((ϕ)(ψ))
∀α[—α—]	minimum depth of α is odd	maximum depth of α is odd
∃α[—α—]	minimum depth of α is even	maximum depth of α is even
α∈β	αβ	
α=β	*I*αβ (eliminable in well-known set theoretic ways). '=' reserved for logical equivalence.	

II. Interpreting the New Syntax.

We now provide an interpretation for our notation and show that every statement in:

- The standard notation for quantification theory is equivalent, on its usual interpretation, to a statement in our notation;
- Our notation is either equivalent to a statement in standard notation, as usually interpreted, or else to an unobjectionable addition to the standard notation.

The interpretation proposed here resembles the usual ones provided for quantification theory, in that variables are assumed to range over some countable domain of individuals. Since our system, like ML, provides for only one kind of variable, no further characterization of the domain is needed except to mention that here, as in ML (Quine, 1951: 121f), the individuals may be classes or abstract entities as well as individuals or concrete entities. <*Pardon my ontological fastidiousness, but what about ML's proper classes, which can appear only to the right of* '$\in$'? *To my knowledge, ML does not allow for individuals* [urelements] *that are neither sets nor classes and that can only appear to the left of* '$\in$'. *Quine dismissed urelements by identifying them with their unit sets.*> As in the usual interpretations, it is also assumed that replacing a variable in an atomic formula with a name results in a statement [*you wrote* "the resulting statement will have two truth values, T and F"]. Moreover, standard truth tables define the meanings of conjunction and denial, and of formulae built from these. Our interpretation differs from standard ones, however, by virtue of:

- How variables are interpreted in any given statement;
- The addition of the name 'E' and its use in place of quantifiers.

In any statement ϕ of our notation, the leftmost occurrence of each variable α is read as "some α", and every subsequent occurrence of α in ϕ is read simply as 'α' and understood to refer back pronominally to the leftmost occurrence. This differs from the usual interpretation of the standard notation where:

- "Some α" is usually associated with the existential quantifier (absent from our notation), while variables are read merely as 'α' or 'it', and are ambiguous names of individuals;
- Variables generally refer back pronominally only to the nearest quantifier on the left containing them, rather than to the leftmost occurrence in the statement.

A consequence of our interpretation is the elimination of propositional functions (open formulae); every formula standing alone is also a statement. Thus:

1. '$x \in y$' means "some x is a member of some y", the standard notation for which is '$(\exists x \exists y(x \in y))$';
2. '$\neg(x \in y)$' means "it is false that some x is a member of some y", or "nothing is a member of anything", the standard notation for which is '$\neg(\exists x \exists y(x \in y))$' or, equivalently, '$(\forall x \forall y \neg(x \in y))$'.

The equivalence of '$x \in y$' and '$\neg(x \in y)$' in our notation to '$(\exists x \exists y(x \in y))$' and '$(\forall x \forall y \neg(x \in y))$' in standard notation is clear by reference to the interpretation given them in any domain. This follows from the customary interpretation of 'some α', namely an alternation of instantiations of α over the individuals of the given domain (assuming a denumerably infinite domain with a suitable rule for permuting names). Thus:

3. '$x \in y$' means '$(1 \in 1 \vee 1 \in 2 \vee 2 \in 1 \vee 2 \in 2 \vee \ldots)$'.
4. '$\neg(x \in y)$' means '$\neg(1 \in 1 \vee 1 \in 2 \vee 2 \in 1 \vee 2 \in 2 \vee \ldots)$'. By DeMorgan's law, this is equivalent to '$(\neg(1 \in 1) \wedge \neg(1 \in 2) \wedge \neg(2 \in 1) \wedge \neg(2 \in 2) \wedge \ldots)$'.

The stipulation that each variable refers back to its first (counting from the left) occurrence in a formula entails that a formula ϕ containing α has one meaning when ϕ stands alone, and another when ϕ is conjoined to another formula ψ also containing α. Thus the meaning of '$(x \in y) \wedge \neg(x \in y)$' is not "Some x is a member of some y and no x is a member of any y", but "Some x is a member of some y and is not a member of that y". The latter, instead of conjoining the meanings of (3) and (4), is an alternation of instantiations, i.e., '$((1 \in 1 \wedge \neg(1 \in 1)) \vee (1 \in 2 \wedge \neg(1 \in 2)) \vee (2 \in 1 \wedge \neg(2 \in 1)) \vee (2 \in 2 \wedge \neg(2 \in 2)) \vee \ldots)$'. The meaning of "Some x is a member of some y and no x is a member of any *y*" requires new variables, to wit ‘$(\neg(x\in y) \wedge \neg(w\in z))$’. In this case, the two formulae are materially equivalent as both are logically false. But the importance of distinguishing the meanings in the interpretation is clear if we compare ‘$x\in y$’ alone with ‘$(\neg(x\in y) \wedge \neg(x\in y))$’ which merely says that "something is not a member of something". In general:

**1. If ϕ and ψ both contain α, $\ulcorner(\phi\wedge\psi)\urcorner$ means an alternation of instantiations of $\ulcorner(\phi\wedge\psi)\urcorner$ over values of α.

Neither our notation nor the standard one enables one to express all we wish to say about an atomic formula solely by means of the formula and its denial. Not only do we want to say ‘$x\in y$’ ("some *x* is a member of some *y*") and ‘$\neg(x\in y)$’ ("no *x* is a member of any *y*"), but also:

5. "Some *x* is a not member of some *y*" or ‘$(\exists x(\exists y\neg(x\in y)))$’
6. "Some *x* is a member of every *y*" or ‘$(\exists x(\forall y(x\in y)))$’
7. "Some *x* is a not member of any *y*" or ‘$(\exists x(\forall y\neg(x\in y)))$’
8. "Some *y* has every *x* as a member" or ‘$(\exists y(\forall x(x\in y)))$’
9. "Some *y* has no *x* a member" or ‘$(\exists y(\forall x\neg(x\in y)))$’

and denials of these. (5) can be expressed, as above, by ‘$(\neg(x\in y) \wedge \neg(x\in y))$’. To provide for (6)–(9), I introduce the name ‘E’, representing the class of things extant. ‘E’ denotes the universal class or predicate in any domain assigned for purposes of interpretation. Thus:

2**. If *a* is an individual in some domain assigned for purposes of interpretation, ‘$a\in\mathrm{E}$’ (or ‘E*a*’) is tautologous, i.e., necessarily has the value T.

Since every sound interpretation of quantification theory requires a domain of *extant* individuals, introducing a name ‘E’ referring to the universal class is unobjectionable, for by the hypothesis of the interpretation, every individual in the chosen domain exists. <*Are you saying any more than Quine's dictum ‘To be is to be the value of a (bound) variable’? Does the emergence of free logic require amending this sentence?*>

By virtue of (1**) and (2**), the standard formulae (6)–(9) can be shown equivalent, under the interpretation, to their counterparts (6′)–(9′) in our notation:

6′. ‘$(\mathrm{E}x \wedge (\forall y(x\in y)))$’
7′. ‘$(\mathrm{E}x \wedge (\forall y\neg(x\in y)))$’
8′. ‘$(\mathrm{E}y \wedge (\forall x(x\in y)))$’
9′. ‘$(\mathrm{E}y \wedge (\forall x\neg(x\in y)))$’

For example, (6) reduces by D8 to ‘$(\mathrm{E}x \wedge \neg(\mathrm{E}y \wedge \neg(x\in y)))$’. By (1**), the latter formula takes the following meaning in the interpretation:

$$((\mathrm{E}1 \wedge (\neg(\mathrm{E}1 \wedge \neg(1\in 1)) \wedge \neg(\mathrm{E}2 \wedge \neg(1\in 2)) \wedge \neg(\mathrm{E}3 \wedge \neg(1\in 3))\ldots)$$
$$\vee (\mathrm{E}2 \wedge (\neg(\mathrm{E}1 \wedge \neg(2\in 1)) \wedge \neg(\mathrm{E}2 \wedge \neg(2\in 2)) \wedge \neg(\mathrm{E}3 \wedge \neg(2\in 3))\ldots)$$
$$\vee (\mathrm{E}3 \wedge (\neg(\mathrm{E}1 \wedge \neg(3\in 1)) \wedge \neg(\mathrm{E}2 \wedge \neg(3\in 2)) \wedge \neg(\mathrm{E}3 \wedge \neg(3\in 3))\ldots) \vee \ldots)$$

Since by (2**), each formula ‘E*a*’ in the interpretation is tautologous, and since dropping a tautology from a conjunction leaves a truth tables unchanged (provided that at least one conjunct remains), the preceding interpretation is equivalent to:

$$((\neg(\neg(1\in 1)) \wedge \neg(\neg(1\in 2)) \wedge \neg(\neg(1\in 3))\ldots) \qquad (((1\in 1) \wedge (1\in 2) \wedge (1\in 3) \wedge \ldots)$$
$$\vee (\neg(\neg(2\in 1)) \wedge \neg(\neg(2\in 2)) \wedge \neg(\neg(2\in 3))\ldots) \qquad \vee ((2\in 1) \wedge (2\in 2) \wedge (2\in 3) \wedge \ldots)$$
$$\vee (\neg(\neg(3\in 1)) \wedge \neg(\neg(3\in 2)) \wedge \neg(\neg(3\in 3))\ldots) \vee \ldots). \qquad \vee ((2\in 1) \wedge (2\in 2) \wedge (2\in 3) \wedge \ldots) \vee \ldots).$$

Eliminating the double negations from the formula on the left yields the formula on the right, one identical to the usual interpretation of '$(\exists x(\forall y(x\in y)))$'. In like manner, each of (7′), (8′), and (9′), as well as their denials, can be shown equivalent to the usual interpretations, given their standard analogues (7), (8), and (9). In general:

3**. Any statement $\ulcorner(\exists\alpha(\phi))\urcorner$ in standard notation is equivalent under the interpretation to the statement $\ulcorner(\mathrm{E}\alpha\phi)\urcorner$ in our notation.

For if ϕ contains α, then in both cases the interpretation is an alternation of instantiations of ϕ, each constructed by replacing α with the name of a different individual. If ϕ does not contain α, then in neither case is the truth table for the interpretation altered by the presence of $\ulcorner(\exists\alpha)\urcorner$ or $\ulcorner(\mathrm{E}\alpha)\urcorner$.

Thus every interpreted formula in standard notation is equivalent to some interpreted formula in our notation. For every formula in standard notation is reducible, by relettering bound variables (and other steps), to an equivalent statement in prenex normal form, i.e., to a form where every quantifier is to the left of every atomic formula bound thereto, and its scope encompasses all expressions to its right. Under these conditions, standard variables are interpreted in precisely the manner assigned to our variables. The prenex normal form may be further reduced via the standard equivalences of $\ulcorner(\forall\alpha(\phi))\urcorner$ and $\ulcorner\neg(\exists\alpha(\neg\phi))\urcorner$, so that it contains only existential quantifiers. By (3**), the meaning of the interpreted statement is unaltered when existential quantifiers are replaced by $\ulcorner(\mathrm{E}\alpha)\urcorner$.

On the other hand, some statements in our notation do not have analogues in standard notation, since $\ulcorner\mathrm{E}\alpha\urcorner$ may occur in ways not permitted to $\ulcorner(\exists\alpha)\urcorner$, namely:

1. $\ulcorner\mathrm{E}\alpha\urcorner$ and $\ulcorner\neg\mathrm{E}\alpha\urcorner$ standing alone are formulae; *<this merges your rules (1) and (4)>*
2. If $\ulcorner\mathrm{E}\alpha\urcorner$ appears anywhere in a conjunction, conjuncts of the form $\ulcorner\mathrm{E}\alpha\urcorner$ do not alter the interpretation of the conjunction;
3. If $\ulcorner\mathrm{E}\alpha\urcorner$ occurs as a conjunct, other instances thereof in the conjunction do not affect the interpretation of the conjunction.

<I am not confident that my wording of (2) *and* (3) *above does justice to your intentions.>*
The class of formulae in our notation having no analogues in standard notation may be characterized roughly as those containing either $\ulcorner\neg\mathrm{E}\alpha\urcorner$ essentially, or instances of $\ulcorner\mathrm{E}\alpha\urcorner$ without any well-formed parts containing α but not 'E'; e.g., '$\neg\mathrm{E}x$', '$\mathrm{E}x$', '$(\forall x(\mathrm{E}x))$', '$(\neg\mathrm{E}x \rightarrow \forall yz(y\in z))$'. But such statements are not objectionable under any standard interpretation, as they merely reflect the hypothesis that every individual in a given domain exists. The following rules enable translating statements in our notation into statements in standard notation and prenex normal form:

1. If the atomic formula ϕ contains α, or is a conjunction at least two of whose conjuncts contain α, then $\ulcorner\mathrm{E}\alpha \wedge \phi\urcorner$ replaces ϕ. (Call this rule *Eα-introduction*);

2. An ⌜Eα⌝ can be moved to the left, either by the associativity of conjunction or by the following rule of passage. If ψ does not contain α, replace ⌜(ψ ∧ ¬(Eα ∧ ¬ϕ))⌝ with ⌜¬(Eα ∧ ¬(ψ∧ϕ))⌝;
3. All except the first occurrences of any ⌜Eα⌝ may then be dropped, since under any interpretation, tautologous conjuncts can be dropped from a conjunction without altering its truth table, provided that at least one conjunct remain.

These rules preserve satisfiability, for all nonempty domains.

III. Concluding Remarks.
Thus far, we have only shown that our second notation adequately expresses the standard meanings of quantified statements, and that statements in our notation lacking counterparts in standard notation are unobjectionable from the point of view of ordinary interpretations. We have neither supplied axioms for our notation, nor should it be assumed that any standard axiom set will do the job. For instance, if Quine's *104 (***modus ponens***) were an axiom in our notation, it would yield false consequents such as '(E*x* ∧ E*y* ∧ *x*∈*y*) → ∀*zw*(*z*∈*w*)' ("If something is a member of something, then everything is a member of everything").

Notwithstanding this limited result, our second notational system suggests a more general system of some possible interest. Begin by assuming that each atomic symbol containing a prime number of o's is reserved for use as some predicate constant, as '(o)' and '(oo)' stand for the predicate constants "—is a member of—" and "—exists". Assume further that any atomic symbol containing a number of o's having the prime *p* as a factor can be interpreted as a predicate containing the associated constant as a conjunctive part. For example, the class of variables, i.e., the class of atomic symbols containing an even number of o's exceeding 2, defines a denumerably infinite collection of symbols, each of which, whatever else it may be, ranges over the domain.

Extensions of first order logic arising from the introduction of new constants, e.g., "—is a class" or "—is a linguistic expression," begin by assigning a distinct atomic symbol to each new constant, such that the number of o's in a new atomic symbol is a prime number *q* differing from the number of o's in any established constant. Variables whose number of o's have *q* as a prime factor may then be used as variables ranging over the entities to which the new predicate applies, e.g., variables for extant linguistic expressions, or extant classes. In this fashion, any conceivable constant can be introduced into our system of notation in a consistent manner. Adding postulates defining the properties of such constants then allows the notation of any applied logic to be incorporated systematically within that of general logic.

References.

Angell, R.B. (1960) "A logical notation with two primitive signs" (abstract), *Journal of Symbolic Logic 25*: 385.

Quine, W.V. (1951) *Mathematical Logic*. Harvard Univ. Press.

Remarks

I value your paper because the notation it proposes is an instance of what Lou Kauffman, I, and a few others call *boundary* notation. If you are willing to go over this paper carefully, correcting this and updating that, I may be willing to include the revised version in a book on boundary methods I am putting together with Lou, a mathematician at Illinois-Chicago. Lou and I could reprint your 1960 *JSL* abstract as is, as evidence of your prescience. Or you could incorporate parts of your abstract into section I.

I have taken the liberty of reorganizing your paper somewhat. I have appended the early paragraphs of your section II to section I. My section III is made up of the last two paragraphs of your section II. I propose wording for the section headings. I have reworked your two syntactical systems to highlight more clearly the distinction between the atomic and the molecular.

Even though I never formally studied logic, and am too young (51) for Quine's *Mathematical Logic* (*ML*) to be my primary source for logic, I am nevertheless a fan of *ML*, having dipped into it over and over again. This, in combination with my independent discovery of what I call *boundary notation*, made section I above an easy and enjoyable read for me. I confess that I do not really follow what you are doing section II. All references to *ML* are now to Quine (1951), the edition Harvard U Press keeps in print. I am very intrigued by *ML*'s set theory, and cannot explain why that theory has no following. (NF and NFU have an ample following in Europe; NFU has an ardent American disciple in Randall Holmes. But Quine's lifelong love for *ML* never found an echo.)

I have made many editorial changes, including shortening some paragraphs and sentences and recasting sentences into the active voice. I have taken the liberty of modernizing your notation for the predicate calculus and some of your terminology. E.g., I have replaced 'wff' with 'formula', 'existing' with 'extant', 'object' with 'individual', 'variable' with 'term', and 'function' with 'functor'. Shortly after you wrote your paper, free logic emerged; hence I have added a mention that the domain is nonempty. Given the date of your paper, I do not rule out further modernizations, such as eliminating the quasi-quotations. I note that the literature has grown relaxed even with respect to regular quotation marks. For that matter, nobody seems to fuss about use versus mention any more.

Your notion of an *atomic symbol* and your notation for variables are new to me. I invite you to consider employing the following notation I've devised. '$\phi\langle\alpha\rangle$' means that at least one instance of 'α' appears in 'ϕ'. Likewise, '$\phi\rangle\alpha\langle$' means that 'ϕ' contains *no* instances of 'α'. A metalogical notation I've devised is '$\phi\Leftrightarrow\gamma$', meaning that '$\phi$' and '$\gamma$' express the same thing, however different their notations. '$\Leftrightarrow$' is very useful when using one notation as a metalanguage to explicate another notation, the object language, a situation with which your paper is rife.

Have you clearly distinguished between p as the prime number of o's appearing in an atomic symbol, on the one hand, and as a prime *factor* of that number, on the other? If your invoking primality here is in any way related to Gödel numbering, please state that fact.

The first boundary notation appeared in two papers C S Peirce wrote in 1886 but not published until 1993. Peirce did not linger over his 1886 notation, instead going on to develop his graphical logic, perhaps the paradigmatic example of boundary notation. Even though the 1931-35 *Collected Papers* devote more than 100pp to Peirce's graphical logic, it was ignored until two fellows wrote Ph.D. theses about it 40 years ago, hence shortly after you wrote your paper. Don Roberts published his thesis results in a 1964 volume of studies on Peirce edited by Moore and Robin, and in a 1973 monograph. Jay Zeman's thesis can be read on his web site.

A spectre haunts modern logic, that of Charles Sanders Peirce. Peirce's beta graphs, devised in the 1890s, point to a dramatic simplification of the mechanics of first order logic, one ignored to date. The beta graphs show that there is no need for explicit quantification. All variables are taken as 'implicitly' quantified. Now determine the minimum depth of a variable in the sense of your paper. If that minimum depth is even [odd], the variable is existentially [universally] quantified. Jay Zeman wrote on this in the 1967 *JSL*, and was probably the first person other than Peirce to understand the beta graphs. Even now, I conjecture that the number of people who truly understand Peirce's graphical logic does not exceed 50, although that number does include Hilary Putnam. I do not include myself among those 50, because I have yet to grasp Peirce's beta graphs to my satisfaction. I am working on a simplification of the beta graphs that would render obsolete natural deduction and refutation trees.

Randall Dipert at SUNY-Buffalo ranks Peirce the logician right behind Aristotle, Frege, Gödel, and Tarski, and I heartily concur. Peirce, not Frege, invented the quantification and first order logic we all teach and love. Peano acknowledged Peirce's influence, and Schröder revered Peirce. Model theorists claim descent from Löwenheim and Skolem, whom Peirce strongly influenced either directly, or indirectly via Schröder. Polish logicians during the 1920s and 30s read Peirce, and Tarski repeatedly noted Peirce's historical importance. Starting from some notation and a dozen theorems by De Morgan, Peirce produced a full-blown algebra of relations that became a backbone of *Principia Mathematica*, **where Peirce is never mentioned**. Russell cites Peirce in his *Principles of Mathematics*, without enthusiasm. I lay the blame for the snubbing of Peirce by several generations of logicians at Russell's and Whitehead's doorstep. (Quine is not to blame; cf. *ML* and *Methods of Logic*.) Whitehead, by the way, expressed mock irritation when Hartshorne showed him Peirce's unpublished writings anticipating a fair bit of W's process metaphysics.

Peirce is the greatest abstract thinker ever to emerge in the western hemisphere. He has long been duly acknowledged as the father of pragmatism. He is also the father of semiotics, but most semioticians march under the banner of Saussure. He wrote much on metaphysics, although his writings under that heading are scattered and sometimes contradictory. For nearly 30 years, Peirce was a working scientist employed by the US government, collecting geodetic and astronomical data. His hands-on experience with empirical science surpasses that of any other philosopher of science, with the possible exception of Pierre Duhem. Feibleman (1946), Goudge (1950) and Murphey (1961) did yeoman's work in trying to survey and assess Peirce's work, but a great deal more work is required. Brent (1993) is the first biography of Peirce; while many Peircians disagree with it, all agree that Peirce's life was a sorry mess. During the last quarter century of his life, he nearly starved and froze to death; such was the cold shoulder American civilization presented to him.

Peirce studies labor under severe textual problems. The *Collected Papers* published by the Harvard Uni. Press 1931-58, is badly flawed, in part because edited by inexperienced scholars, namely Hartshorne and Weiss at the dawn of their careers. The Peirce *Nachlass* is perhaps the most daunting one in our civilization. The circa 80,000 ms pages found in his study at his death were not catalogued and microfilmed until 1971. Thirty years ago, a team funded by the NEH began working on a critical edition. To date, only 6 of the projected 30 volumes have appeared.